Supporting Information for
"Revisiting ENSO/Indian Ocean Dipole phase relationships"

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Text S1: Full derivation of the approximate solution for the deterministic IOD model

Here we are going to show that the observed spectral and crosscorrelation characteristics of the IOD can be explained by the timescales generated by our simple deterministic IOD model (equation 3 without noise). For simplicity, we assume a sinusoidal ENSO forcing with frequency ω_0 and phase φ_0 to simplify equation (3), and use the initial condition $T(0) = T_0$:

$$\frac{dT}{dt} = T'(t) = -[\lambda_0 - \lambda_a \cos(\omega_a t + \varphi_l)]T(t) - \cos(\omega_0 t + \varphi_0)\cos(\omega_a t + \varphi_a). \tag{1}$$

The standard form of this inhomogeneous linear first order ordinary differential equation with a variable coefficient is:

$$T'(t) + p(t)T(t) = q(t), \tag{2}$$

with $p(t) = \lambda_0 - \lambda_a \cos(\omega_a t + \varphi_l)$ and $q(t) = -\cos(\omega_0 t + \varphi_0)\cos(\omega_a t + \varphi_a)$.

For which the following general solution exists:

$$T(t) = \frac{1}{u(t)}\left[\int u(t)q(t)dt + C\right], \tag{3}$$

with the integration factor $u(t) = e^{\int p(t)dt}$. The analytical solution of the initial value problem (1) can be written as:

$$T(t) = -e^{-\lambda_0 t + \frac{\lambda_a}{\omega_a}\sin(\omega_a t + \varphi_l)}\left[\int e^{\lambda_0 t - \frac{\lambda_a}{\omega_a}\sin(\omega_a t + \varphi_l)}\cos(\omega_a t + \varphi_a)\cos(\omega_0 t + \varphi_0)\,dt\right] + Ce^{-\lambda_0 t + \frac{\lambda_a}{\omega_a}\sin(\omega_a t + \varphi_l)}. \tag{4}$$

The second term $Ce^{-\lambda_0 t + \frac{\lambda_a}{\omega_a}\sin(\omega_a t + \varphi_l)}$ vanishes as the time t increases. Here, we shall focus on the first term, as it has information about the unfolding of the combination tones. Let T_s be the first term. Therefore,

$$T_s(t) = -e^{-\lambda_0 t}e^{\frac{\lambda_a}{\omega_a}\sin(\omega_a t + \varphi_l)}\left[\int e^{\lambda_0 t}e^{-\frac{\lambda_a}{\omega_a}\sin(\omega_a t + \varphi_l)}\cos(\omega_a t + \varphi_a)\cos(\omega_0 t + \varphi_0)\,dt\right]. \tag{5}$$

As $\epsilon = \frac{\lambda_a}{\omega_a} < 1$ (which is the case because $\lambda_a \leq 1/4\text{month}^{-1}$ and $\omega_a = 2\pi/12$ month^{-1}), we can use a Taylor expansion (0th and 1st order) of the exponential functions $e^{f(t)} = 1 + \epsilon f(t) + O(\epsilon^2)$, with $f(t) = \frac{\lambda_a}{\omega_a}\sin(\omega_a t + \varphi_l)$, to write $\tilde{T}$, the approximation of T_s:

$$\tilde{T}(t) = -e^{-\lambda_0 t}\ [1 + \epsilon \sin(\omega_a t + \varphi_l)] \left[\int e^{\lambda_0 t}\ [1 - \epsilon \sin(\omega_a t + \varphi_l)] \cos(\omega_a t + \varphi_a) \cos(\omega_0 t + \varphi_0)\ dt \right]. \tag{6}$$

Which can be written in the following form:

$$\begin{aligned} \tilde{T}(t) = -e^{-\lambda_0 t}\ [1 + \epsilon \sin(\omega_a t + \varphi_l)] \\ \{ \int e^{\lambda_0 t} \cos(\omega_a t + \varphi_a) \cos(\omega_0 t + \varphi_0) dt \\ - \int \frac{\epsilon e^{\lambda_0 t}}{4} \{ \sin[(2\omega_a - \omega_0)t + 2\varphi_a - \varphi_0] + \sin[(2\omega_a + \omega_0)t + 2\varphi_a + \varphi_0] \} dt \}. \end{aligned} \tag{7}$$

Working out the integration, we obtain:

$$\begin{aligned} \tilde{T}(t) = -e^{-\lambda_0 t}\ [1 + \epsilon \sin(\omega_a t + \varphi_l)]\ \{ \\ \frac{e^{\lambda_0 t}}{2} \{ \frac{\lambda_0 \cos[(\omega_a - \omega_0)t + \varphi_a - \varphi_0] + (\omega_a - \omega_0) \sin[(\omega_a - \omega_0)t + \varphi_a - \varphi_0]}{\lambda_0^2 + (\omega_0 - \omega_a)^2} \\ + \frac{\lambda_0 \cos[(\omega_a + \omega_0)t + \varphi_a + \varphi_0] + (\omega_a + \omega_0) \sin[(\omega_a + \omega_0)t + \varphi_a + \varphi_0]}{\lambda_0^2 + (\omega_0 + \omega_a)^2} \} \\ + \frac{\epsilon e^{\lambda_0 t}}{4} \{ \frac{\lambda_0 \sin[(2\omega_a - \omega_0)t + 2\varphi_a - \varphi_0] + (2\omega_a - \omega_0) \cos[(2\omega_a - \omega_0)t + 2\varphi_a - \varphi_0]}{\lambda_0^2 + (\omega_0 - 2\omega_a)^2} \\ + \frac{\lambda_0 \sin[(2\omega_a + \omega_0)t + 2\varphi_a + \varphi_0] + (2\omega_a + \omega_0) \cos[(2\omega_a + \omega_0)t + 2\varphi_a + \varphi_0]}{\lambda_0^2 + (\omega_0 + 2\omega_a)^2} \} \\ \}. \end{aligned} \tag{8}$$

Defining the following phase angles $\cos\theta_0 = \frac{\lambda_0}{\sqrt{\lambda_0^2 + (\omega_a - \omega_0)^2}}$,

$\sin\theta_0 = \frac{\omega_a - \omega_0}{\sqrt{\lambda_0^2 + (\omega_a - \omega_0)^2}}$,

$\cos\theta_0^* = \frac{\lambda_0}{\sqrt{\lambda_0^2 + (\omega_0 + \omega_a)^2}}$,

$\sin\theta_0^* = \frac{\omega_a + \omega_0}{\sqrt{\lambda_0^2 + (\omega_0 + \omega_a)^2}}$,

$\cos\theta_1 = \frac{\lambda_0}{\sqrt{\lambda_0^2 + (2\omega_a - \omega_0)^2}}$,

$\sin\theta_1 = \frac{2\omega_a - \omega_0}{\sqrt{\lambda_0^2 + (2\omega_a - \omega_0)^2}}$,

$\cos\theta_1^* = \frac{\lambda_0}{\sqrt{\lambda_0^2 + (\omega_0 + 2\omega_a)^2}}$, and

$\sin\theta_1^* = \frac{2\omega_a + \omega_0}{\sqrt{\lambda_0^2 + (\omega_0 + 2\omega_a)^2}}$

simplifies the equation to:

$$
\begin{aligned}
\tilde{T}(t) = -e^{-\lambda_0 t}\ [1 + \epsilon \sin(\omega_a t + \varphi_l)]\ \{ \\
\frac{e^{\lambda_0 t}}{2\sqrt{\lambda_0^2 + (\omega_a - \omega_0)^2}} \{\cos[(\omega_a - \omega_0)t + \varphi_a - \varphi_0] \cos\theta_0 + \sin\theta_0 \sin[(\omega_a - \omega_0)t + \varphi_a - \varphi_0]\} \\
+ \frac{e^{\lambda_0 t}}{2\sqrt{\lambda_0^2 + (\omega_a + \omega_0)^2}} \left\{\cos[(\omega_a + \omega_0)t + \varphi_a + \varphi_0] \cos\theta_0^* + \sin\theta_0^* \sin[(\omega_a + \omega_0)t + \varphi_a + \varphi_0]\right\} \\
+ \frac{\epsilon e^{\lambda_0 t}}{4\sqrt{\lambda_0^2 + (2\omega_a - \omega_0)^2}} \{\sin[(2\omega_a - \omega_0)t + 2\varphi_a - \varphi_0] \cos\theta_1 + \sin\theta_1 \cos[(2\omega_a - \omega_0)t + 2\varphi_a - \varphi_0]\} \\
+ \frac{\epsilon e^{\lambda_0 t}}{4\sqrt{\lambda_0^2 + (2\omega_a + \omega_0)^2}} \left\{\sin[(2\omega_a + \omega_0)t + 2\varphi_a + \varphi_0] \cos\theta_1^* + \sin\theta_1^* \cos[(2\omega_a + \omega_0)t + 2\varphi_a + \varphi_0]\right\}\}.
\end{aligned}
\tag{9}
$$

Which can be combined to:

$$
\begin{aligned}
\tilde{T}(t) = -e^{-\lambda_0 t}\ [1 + \epsilon \sin(\omega_a t + \varphi_l)]\ \{ \\
\frac{e^{\lambda_0 t}}{2\sqrt{\lambda_0^2 + (\omega_a - \omega_0)^2}} \{\cos[(\omega_a - \omega_0)t - \theta_0 + \varphi_a - \varphi_0]\} \\
+ \frac{e^{\lambda_0 t}}{2\sqrt{\lambda_0^2 + (\omega_a + \omega_0)^2}} \left\{\cos[(\omega_a + \omega_0)t - \theta_0^* + \varphi_a + \varphi_0]\right\} \\
+ \frac{\epsilon e^{\lambda_0 t}}{4\sqrt{\lambda_0^2 + (2\omega_a - \omega_0)^2}} \{\sin[(2\omega_a - \omega_0)t + \theta_1 + 2\varphi_a - \varphi_0]\} \\
+ \frac{\epsilon e^{\lambda_0 t}}{4\sqrt{\lambda_0^2 + (2\omega_a + \omega_0)^2}} \left\{\sin[(2\omega_a + \omega_0)t + \theta_1^* + 2\varphi_a + \varphi_0]\right\}\}.
\end{aligned}
\tag{10}
$$

As $\omega_0 < \omega_a$, the amplitude factors for the sum ($\omega_a + \omega_0$ and $2\omega_a + \omega_0$) and difference tones ($\omega_a - \omega_0$ and $2\omega_a - \omega_0$) can be written in the following form with only small error for realistic parameters:

$$
\begin{aligned}
\tilde{T}(t) = -e^{-\lambda_0 t}\ [1 + \epsilon \sin(\omega_a t + \varphi_l)]\ \{ \\
\frac{e^{\lambda_0 t}}{\sqrt{\lambda_0^2 + \omega_a^2}} \left[\cos(\omega_a t - \theta_0^{\#} + \varphi_a) \cos(\omega_0 t + \varphi_0)\right] \\
+ \frac{\epsilon e^{\lambda_0 t}}{2\sqrt{\lambda_0^2 + 2\omega_a^2}} \left[\sin(2\omega_a t + \theta_1^{\#} + 2\varphi_a) \cos(\omega_0 t + \varphi_0)\right]\},
\end{aligned}
\tag{11}
$$

where we define $\theta_0^\# = (\theta_0 + \theta_0^*)/2$ and $\theta_1^\# = (\theta_1 + \theta_1^*)/2$.

Finally, we get a separate term for each timescale (neglecting any terms of order $O(\epsilon^2)$):

$$
\begin{aligned}
\tilde{T}(t) = &-\overbrace{\frac{1}{\sqrt{\lambda_0^2 + \omega_a^2}} \left[\cos(\omega_a t - \theta_0^\# + \varphi_a)\cos(\omega_0 t + \varphi_0)\right]}^{1 \pm f_E} \\
&-\overbrace{\frac{\epsilon}{2\sqrt{\lambda_0^2 + \omega_a^2}} \left[\sin(\varphi_l - \varphi_a + \theta_0^\#)\cos(\omega_0 t + \varphi_0)\right]}^{f_E} \\
&-\overbrace{\frac{\epsilon}{2\sqrt{\lambda_0^2 + \omega_a^2}} \left[\sin(2\omega_a t + \varphi_a + \varphi_l - \theta_0^\#)\cos(\omega_0 t + \varphi_0)\right]}^{2 \pm f_E} \\
&-\overbrace{\frac{\epsilon}{2\sqrt{\lambda_0^2 + 2\omega_a^2}} \left[\sin(2\omega_a t + \theta_1^\# + 2\varphi_a)\cos(\omega_0 t + \varphi_0)\right]}^{2 \pm f_E}.
\end{aligned} \tag{12}
$$

Hence, the seasonally modulated damping rate is responsible for the frequency unfolding from the C-mode forcing ($1 \pm f_E$) to a response that includes both the ENSO timescale (f_E) and the semi-annual combination tones ($2 \pm f_E$) additional to the C-mode timescale. The neglected higher order terms of the Taylor series expansion would further correct the amplitudes of the existing terms as well as create additional higher frequency terms (such as $3 \pm f_E$). The error estimate as a function of λ_a is shown in Fig. S4.

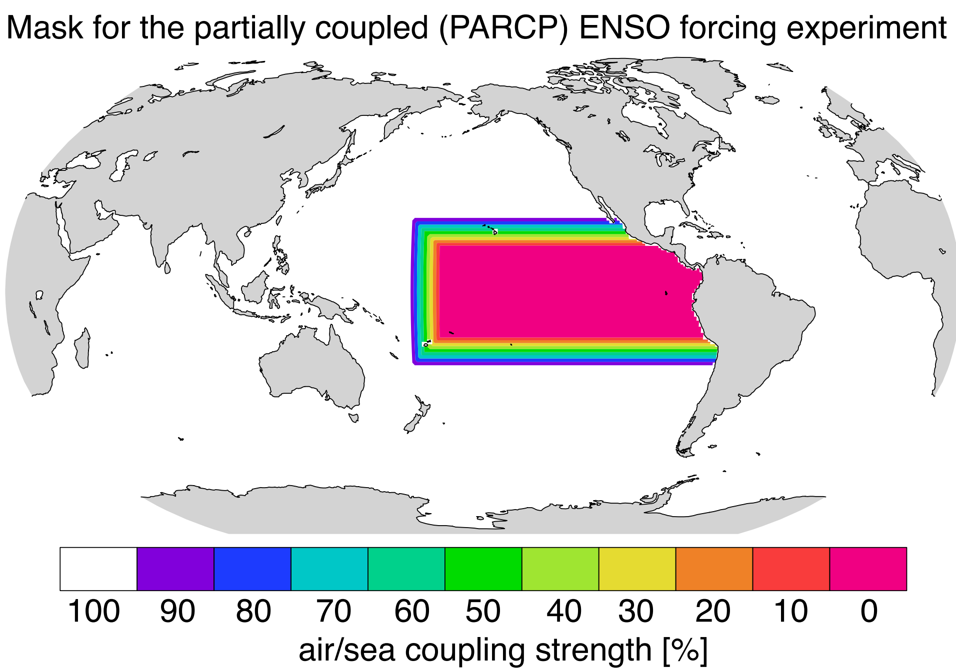

Figure S1. Mask for the CM2.1 partially coupled (PARCP) experiments with prescribed ENSO forcing. Shading indicates the air/sea coupling strength [%]. A 5 day damping time scale to prescribed SSTs is used in the shaded areas.

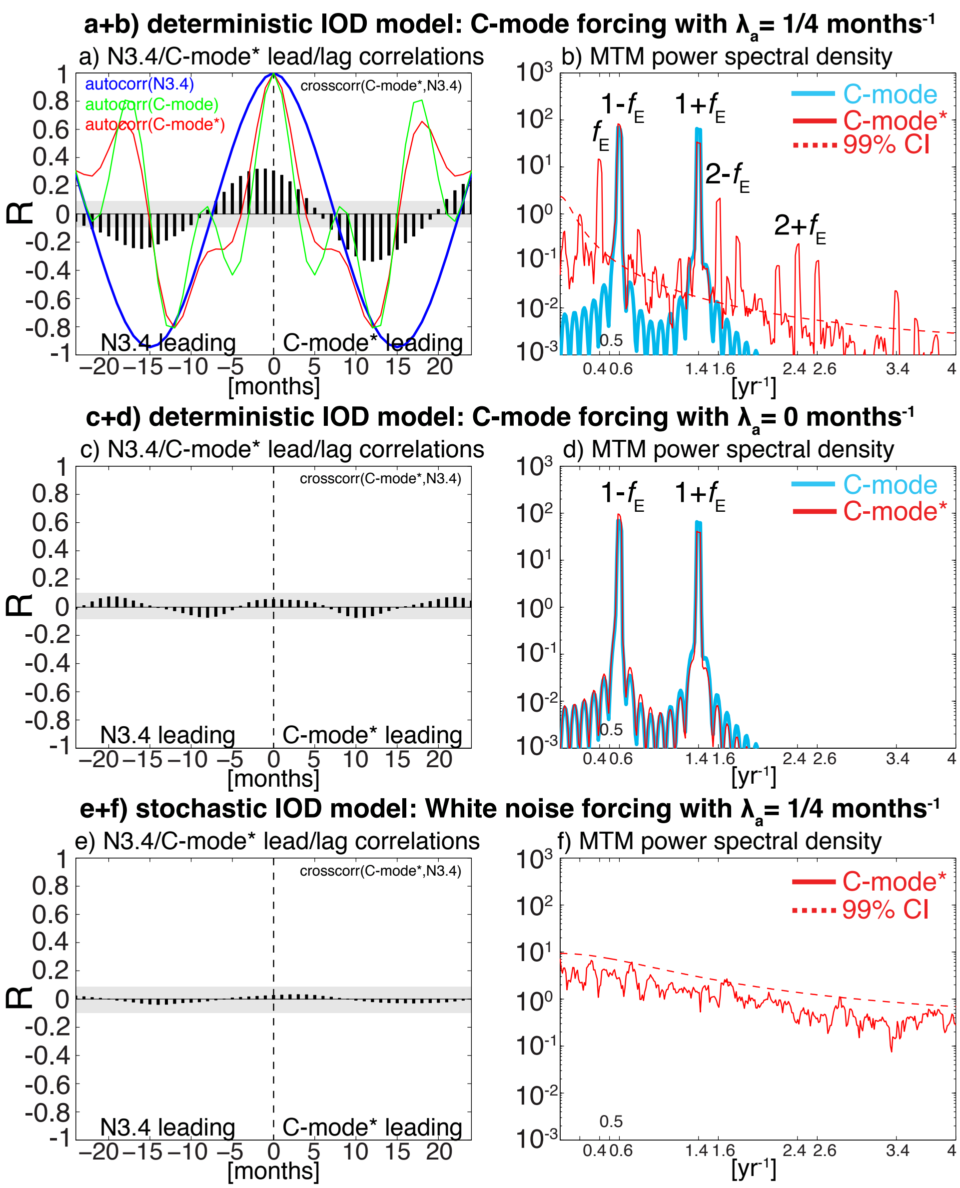

Figure S2. Lead/lag crosscorrelations (left column) and MTM power spectra (right column) for various simple IOD models (each solution labeled as C-mode*) for a C-mode with a sinusoidal ENSO period of 2.5 years. First row: Deterministic IOD model with C-mode forcing and $\lambda_a = 1/4$ (as used throughout the paper). Second row: The same model but without a seasonally varying damping rate ($\lambda_a = 0$). Third row: Stochastic IOD model (equation 2 in the main manuscript) with white noise forcing (no C-mode forcing and hence no ENSO information) and a seasonally varying damping rate ($\lambda_a = 1/4$). (**a**) Lead/lag correlations between monthly N3.4 and C-mode* (black bars), as well as the autocorrelations for N3.4 (blue line), C-mode (green line), and C-mode* (red line). (**c,e**) Lead/lag correlations between monthly N3.4 and C-mode* (black bars). (**b,d**) MTM power spectra for the C-mode forcing (cyan line) and C-mode* (red line). (**f**) MTM power spectrum for C-mode* with white noise forcing and seasonally modulated damping rate (red spectrum).

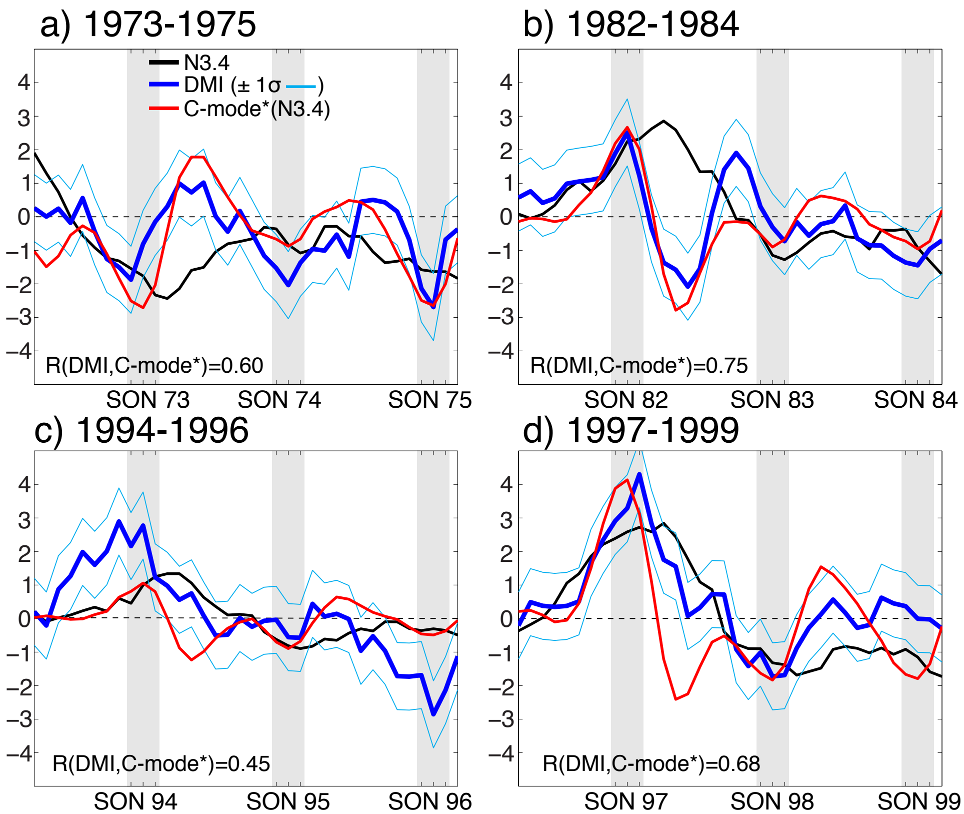

Figure S3. Time evolution of the normalized monthly N3.4 (black line) and DMI (blue line) indices for (**a**) several consecutive La Niña events, (**b**) the large 1982/83 El Niño, (**c**) the 1994 IOD, and (**d**) the large 1997/98 El Niño. As a measure of the DMI variability, the thin cyan lines indicate the one standard deviation spread. The integrated combination mode reconstruction (C-mode*) is based on the N3.4 index (equation 3). The linear correlation (R) between the reconstruction and the DMI in each time segment is given as inserts.

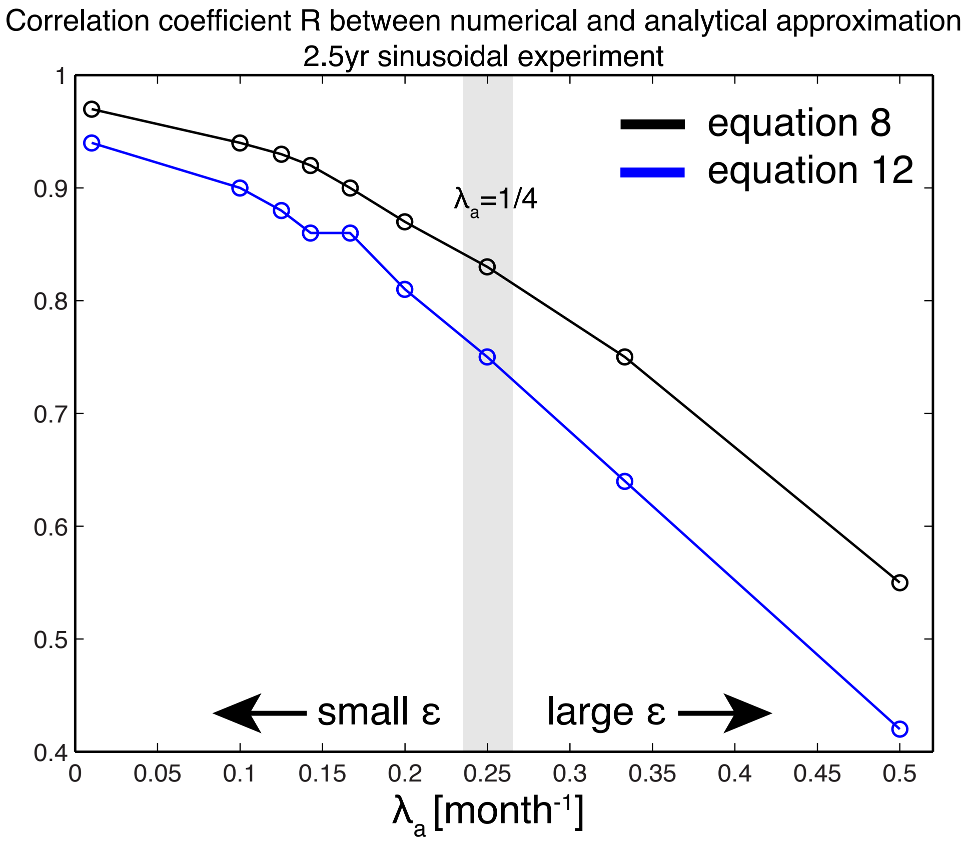

Figure S4. Error estimate for the Taylor expansion truncation by comparing the numerical result for the C-mode* with the analytical approximations.